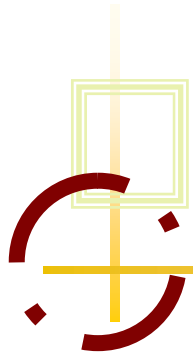


8. Line transfer

The two-level atom

Milne-Eddington model

curve of growth



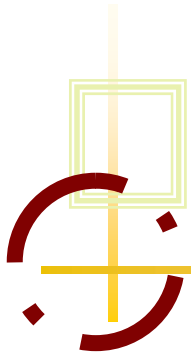
Two-level approximation

We consider schematic line-formation cases with easy solution

highly simplified: not accurate, but provide insight into the mechanisms at work in real stellar atmospheres.

Consider an atomic model with only two important levels: lower l and upper u .

Although highly simplified, it well approximates the situation for some lines, *e.g.* resonance lines from the ground state.



Two-level approximation

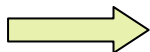
Rate equations (statistical equilibrium)

$$n_i \sum_{j \neq i} (R_{ij} + C_{ij}) + n_i (R_{ik} + C_{ik}) = \sum_{j \neq i} n_j (R_{ji} + C_{ji}) + n_p (R_{ki} + C_{ki})$$

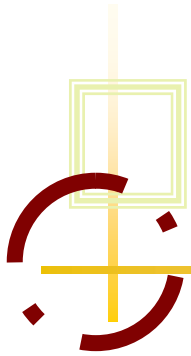
Consider two levels u and l : isolating the transitions between them

$$n_l (R_{lu} + C_{lu}) + n_l \sum_{j \neq l, u} (R_{lj} + C_{lj}) + n_l (R_{lk} + C_{lk}) = n_u (R_{ul} + C_{ul}) + \sum_{j \neq l, u} n_j (R_{jl} + C_{jl}) + n_p (R_{kl} + C_{kl})$$

and neglecting all transitions involving $j \neq l, u$, plus recombinations/ionizations:



$$n_l (R_{lu} + C_{lu}) = n_u (R_{ul} + C_{ul})$$



Two-level approximation

substituting for the R coefficients:

$$n_l \left(B_{lu} \int_0^\infty \varphi_\nu J_\nu d\nu + C_{lu} \right) = n_u \left(A_{ul} + B_{ul} \int_0^\infty \varphi_\nu J_\nu d\nu + C_{ul} \right)$$

assuming collision rates dominate over radiative rates

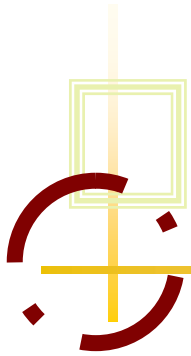
$$n_l C_{lu} = n_u C_{ul}$$

remembering that

$$C_{lu} = \left(\frac{n_u}{n_l} \right)^* C_{ul} = \frac{g_u}{g_l} e^{-E_{ul}/kT} C_{ul}$$

$$\Rightarrow n_l \frac{g_u}{g_l} e^{-E_{ul}/kT} C_{ul} = n_u C_{ul}$$

$$\Rightarrow \frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-E_{ul}/kT} = \left(\frac{n_u}{n_l} \right)^{\text{LTE}}$$



Two-level approximation

Calculation of the line source function

$$\kappa_{\nu}^{\text{line}} = (n_l B_{lu} - n_u B_{ul}) \varphi_{\nu} \frac{h\nu}{4\pi}$$

$$\epsilon_{\nu}^{\text{line}} = n_u A_{ul} \varphi_{\nu} \frac{h\nu}{4\pi}$$

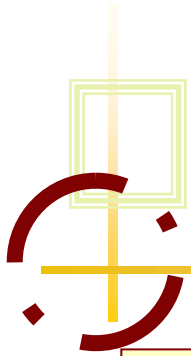


$$S_{\nu}^{\text{line}} = \frac{\epsilon_{\nu}^{\text{line}}}{\kappa_{\nu}^{\text{line}}} = \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}} = \frac{A_{ul}}{\frac{n_l}{n_u} B_{lu} - B_{ul}}$$

with $B_{lu} = \frac{g_u}{g_l} B_{ul}$ $A_{ul} = \frac{2h\nu^3}{c^2} B_{ul}$

$$S_{\nu}^{\text{line}} = \frac{2h\nu^3}{c^2} \frac{1}{\frac{n_l g_u}{n_u g_l} - 1}$$

Note: this is the general expression for the line source function in NLTE. It is always valid (not only in 2-level approximation). What is different in the general case, is how n_l , n_u are computed



Two-level approximation

If we substitute $\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-E_{ul}/kT} = \left(\frac{n_u}{n_l}\right)^*$ $E = h\nu$

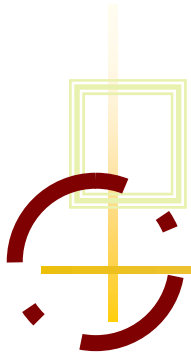
LTE

we recover the Planck function $S_\nu^{\text{line}} = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} = B_\nu(T)$

For the 2-level atom we found $n_l(B_{lu} \int_0^\infty \varphi_\nu J_\nu d\nu + C_{lu}) = n_u(A_{ul} + B_{ul} \int_0^\infty \varphi_\nu J_\nu d\nu + C_{ul})$

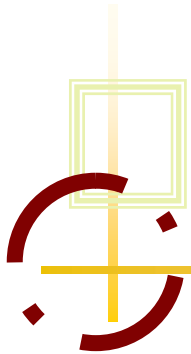
$\Rightarrow \frac{n_u}{n_l} = \frac{1 + \frac{c^2}{2h\nu^3} \int \varphi_{\nu'} J_{\nu'} d\nu' + C_{ul}/A_{ul}}{\frac{g_u}{g_l} \left[\frac{c^2}{2h\nu^3} \int \varphi_{\nu'} J_{\nu'} d\nu' + e^{-h\nu/kT} C_{ul}/A_{ul} \right]}$

and substituting in S_ν :



Two-level approximation

$$\begin{aligned}
 S_{\nu}^{\text{line}} &= \frac{2h\nu^3}{c^2} \frac{\frac{c^2}{2h\nu^3} \int \varphi_{\nu'} J_{\nu'} d\nu' + e^{-h\nu/kT} C_{ul}/A_{ul}}{1 + \frac{C_{ul}}{A_{ul}} (1 - e^{-h\nu/kT})} = \\
 &= \underbrace{\frac{1}{1 + \frac{C_{ul}}{A_{ul}} (1 - e^{-h\nu/kT})}}_{1 - \varepsilon} \int \varphi_{\nu'} J_{\nu'} d\nu' + \underbrace{\frac{2h\nu^3}{c^2} \frac{e^{-h\nu/kT} C_{ul}/A_{ul}}{1 + \frac{C_{ul}}{A_{ul}} (1 - e^{-h\nu/kT})}}_{\frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \underbrace{\frac{(1 - e^{-h\nu/kT}) C_{ul}/A_{ul}}{1 + \frac{C_{ul}}{A_{ul}} (1 - e^{-h\nu/kT})}}_{:= \varepsilon}}_{B_{\nu}(T)}
 \end{aligned}$$



Two-level approximation

$$S_{\nu}^{\text{line}} = (1 - \epsilon) \int_0^{\infty} \varphi_{\nu'} J_{\nu'} d\nu' + \epsilon B_{\nu}(T) = \frac{\int_0^{\infty} \varphi_{\nu'} J_{\nu'} d\nu' + \epsilon' B_{\nu}(T)}{1 + \epsilon'}$$

scattering term
thermal term

$$\epsilon := \frac{(1 - e^{-h\nu/kT}) C_{ul}/A_{ul}}{1 + \frac{C_{ul}}{A_{ul}} (1 - e^{-h\nu/kT})} = \frac{\epsilon'}{1 + \epsilon'}$$

destruction probability

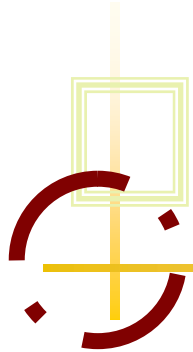
photons are either destroyed into thermal pool or scattered without change in frequency and isotropically

photons are created in thermal processes (ϵB_{ν})

Comparison with Chapter 3:
Continuum source function with true absorption + scattering

$$S_{\nu} = \frac{\kappa_{\nu}}{\kappa_{\nu} + \sigma_{\nu}} B_{\nu} + \frac{\sigma_{\nu}}{\kappa_{\nu} + \sigma_{\nu}} J_{\nu}$$

→ line source function has similar terms except that we also allow for non-coherent scattering

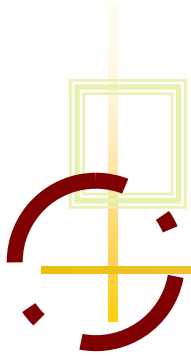


Two-level approximation

deep layers: collisions dominate $\rightarrow \varepsilon' \gg 1$ or $\varepsilon = 1$ *thermal* term dominant
higher layers: collisions non-important $\rightarrow \varepsilon' \approx 0$ $\varepsilon = 0$ *scattering* term dominant

- | | | | | |
|------------------------|-------------------------|---------------|--|-----------------|
| a) $C_{ul} \gg A_{ul}$ | $\varepsilon = 1$ | \rightarrow | $S_\nu = B_\nu(T)$ | LTE |
| b) $C_{ul} \ll A_{ul}$ | $\varepsilon \approx 0$ | \rightarrow | $S_\nu = \int \varphi_{\nu'} J_{\nu'} d\nu'$ | extreme non-LTE |

Chapter 3: $S_\nu = J_\nu$ for pure *coherent* scattering
now $S_\nu = \int \varphi_{\nu'} J_{\nu'} d\nu'$ *non-coherent* scattering



Two-level approximation

This second case ($S_\nu \neq B_\nu$) has a macroscopic interpretation in terms of *scattering*

$$S_\nu^{\text{line}} = \frac{\epsilon_\nu^{\text{line}}}{\kappa_\nu^{\text{line}}} = \int_0^\infty \varphi_{\nu'} J_{\nu'} d\nu' \quad \Longrightarrow \quad \epsilon_\nu^{\text{line}} = \kappa_\nu^{\text{line}} \int_0^\infty \varphi_{\nu'} J_{\nu'} d\nu'$$

for a narrow absorption profile function $\varphi_{\nu'} \approx \delta(\nu' - \nu)$

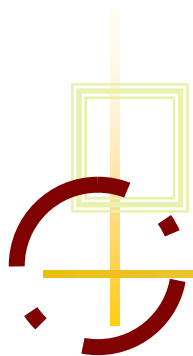


$$\epsilon_\nu^{\text{line}} = \kappa_\nu^{\text{line}} J_\nu$$

as in a coherent scattering process (e.g. Thomson scattering)
complete redistribution: emission and absorption profiles identical

microscopically there is a photon absorption $l \rightarrow u$ followed by re-emission

2-level atom is a special NLTE case
in general the coupling between J_ν , n_i and S_ν is far more complicated



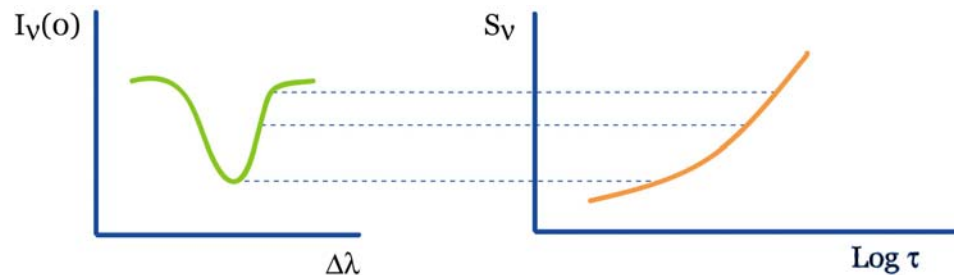
Two-level approximation

Moving outward in the photosphere scattering term dominates

At some point we reach the region where photons are being lost from the star (small optical depth)

→ J_ν decreases with height → S_ν decreases with height

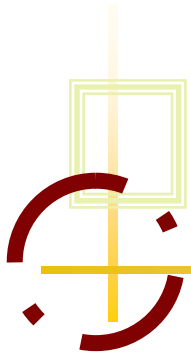
→ *absorption line*



mapping between source function (decreasing outward) and line profile

line absorption coefficient larger at line center → see higher layers

wings formed in deeper layers than line core



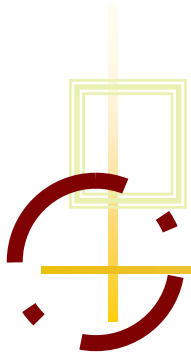
The Milne-Eddington model

We have defined line and continuum coefficients for absorption and emission

Total absorption coefficient is $\kappa_\nu = \kappa_\nu^C + \kappa_\nu^L + \sigma$

with σ from scattering in the continuum

The *line* optical depth is $d\tau_\nu = (\kappa_\nu^C + \kappa_\nu^L + \sigma) dx$
(larger than in the *continuum*!)



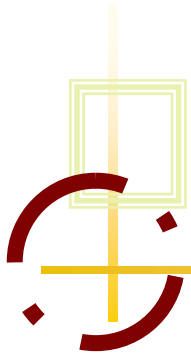
The Milne-Eddington model

Transfer equation:

$$\mu \frac{dI_\nu(\mu)}{dx} = (\kappa_\nu^C + \kappa_\nu^L + \sigma) I_\nu - \epsilon_\nu^C - \epsilon_\nu^L - \sigma J_\nu$$

Without dealing with the general case for the computation of all coefficients we assume:

- LTE in the continuum $\frac{\epsilon_\nu^C}{\kappa_\nu^C} = B_\nu(T)$
- scattering negligible in the continuum $\sigma \ll \kappa_\nu^C$
- 2-level atom in the line

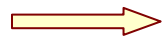


The Milne-Eddington model

$$\mu \frac{dI_\nu}{dx} = (\kappa_\nu^C + \kappa_\nu^L) \left[I_\nu - \frac{\kappa_\nu^C}{\kappa_\nu^C + \kappa_\nu^L} B_\nu - \frac{\kappa_\nu^L}{\kappa_\nu^C + \kappa_\nu^L} S_\nu^L \right]$$

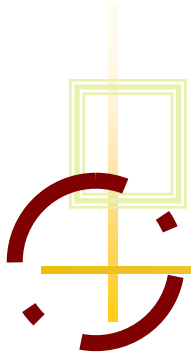
with $\beta_\nu = \frac{\kappa_\nu^L}{\kappa_\nu^C}$ $d\tau_\nu = (\kappa_\nu^C + \kappa_\nu^L) dx = \kappa_\nu^C (1 + \beta_\nu) dx$

$$\lambda_\nu = \frac{1 + \epsilon\beta_\nu}{1 + \beta_\nu} \quad \text{with } \epsilon \text{ from } S_\nu^{\text{line}} = (1 - \epsilon) \int_0^\infty \varphi_{\nu'} J_{\nu'} d\nu' + \epsilon B_\nu(T)$$



$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \lambda_\nu B_\nu - (1 - \lambda_\nu) \int_0^\infty \varphi_\nu J_\nu d\nu$$

Integro-differential equation



The Milne-Eddington model

assumptions (for analytical solution)

1. λ_ν , ε and β_ν constant with depth
2. B_ν linear in continuum optical depth

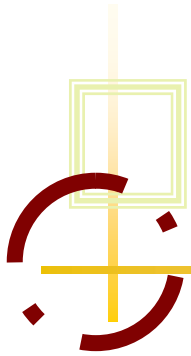
$$B_\nu = a + b \tau_c$$
$$d\tau_c = \frac{d\tau_\nu}{1 + \beta_\nu} \quad \tau_c = \frac{\tau_\nu}{1 + \beta_\nu}$$

3.
$$\int_0^\infty \varphi_\nu J_\nu d\nu \approx J_\nu$$

plus the Eddington approximation

$$K_\nu = \frac{1}{2} \int_{-1}^1 I_\nu \mu^2 d\mu = \frac{1}{3} J_\nu$$

$$H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0) \quad \text{boundary condition}$$



The Milne-Eddington model

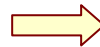
$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \lambda_\nu B_\nu - (1 - \lambda_\nu) J_\nu$$



$$\frac{1}{2} \int_{-1}^1 \dots d\mu : \quad \frac{dH_\nu}{d\tau_\nu} = J_\nu - \lambda_\nu B_\nu - (1 - \lambda_\nu) J_\nu$$

$$\frac{1}{2} \int_{-1}^1 \dots \mu d\mu : \quad \frac{dK_\nu}{d\tau_\nu} = H_\nu$$

differentiate this equation again



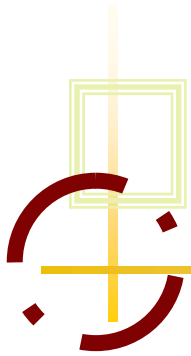
$$\frac{d^2 K_\nu}{d\tau_\nu^2} = J_\nu - \lambda_\nu B_\nu - (1 - \lambda_\nu) J_\nu$$



Eddington approximation

$$\frac{1}{3} \frac{d^2 J_\nu}{d\tau_\nu^2} = \lambda_\nu (J_\nu - B_\nu)$$

B_ν is linear in τ : $\frac{1}{3} \frac{d^2 B_\nu}{d\tau_\nu^2} = 0$



The Milne-Eddington model

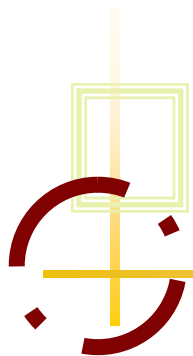
$$\Rightarrow \frac{1}{3} \frac{d^2}{d\tau_\nu^2} (J_\nu - B_\nu) = \lambda_\nu (J_\nu - B_\nu)$$

solution: $J_\nu - B_\nu = \alpha_\nu e^{-\sqrt{3\lambda_\nu} \tau_\nu} + \gamma_\nu e^{\sqrt{3\lambda_\nu} \tau_\nu}$

boundary conditions $\tau_\nu \rightarrow \infty : J_\nu \rightarrow B_\nu \rightarrow \gamma_\nu = 0$

$$\tau_\nu = 0 : H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0)$$

we use this boundary condition to determine α_ν on the next 2 pages



The Milne-Eddington model

from $\frac{dK_\nu}{d\tau_\nu} = \frac{1}{3} \frac{dJ_\nu}{d\tau_\nu} = H_\nu \quad \frac{1}{3} \frac{dJ_\nu}{d\tau_\nu} \Big|_{\tau_\nu=0} = H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0)$

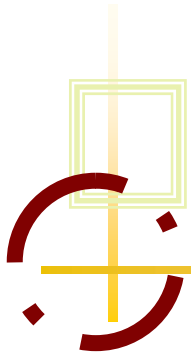
→ $J_\nu(0) = \alpha_\nu + B_\nu(0) = \alpha_\nu + a = \frac{1}{\sqrt{3}} \frac{dJ_\nu}{d\tau_\nu} \Big|_{\tau_\nu=0}$

from $J_\nu - B_\nu = \alpha_\nu e^{-\sqrt{3\lambda_\nu} \tau_\nu} + \gamma_\nu e^{\sqrt{3\lambda_\nu} \tau_\nu}$

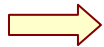
and $B_\nu = a + b\tau_c$

$$\frac{1}{\sqrt{3}} \frac{dJ_\nu}{d\tau_\nu} \Big|_{\tau_\nu=0} = \frac{1}{\sqrt{3}} \left[-\alpha_\nu \sqrt{3\lambda_\nu} + \frac{b}{1 + \beta_\nu} \right] = \alpha_\nu + a$$

$$\frac{dB_\nu}{d\tau_\nu} = \frac{dB_\nu}{d\tau_c} \frac{1}{1 + \beta_\nu}$$



The Milne-Eddington model



$$\alpha_\nu = \frac{\frac{b}{1+\beta_\nu} - \sqrt{3}a}{(\sqrt{3} + \sqrt{3\lambda_\nu})}$$

define

$$p_\nu = \frac{b}{1 + \beta_\nu}$$

$$J_\nu(\tau) = a + p_\nu \tau_\nu + \frac{p_\nu - a\sqrt{3}}{\sqrt{3} + \sqrt{3\lambda_\nu}} e^{-\sqrt{3\lambda_\nu} \tau_\nu}$$

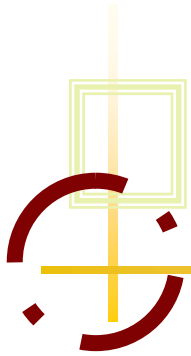
$$H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0) = \frac{a}{\sqrt{3}} + \frac{p_\nu - a\sqrt{3}}{3(1 + \sqrt{\lambda_\nu})} = \frac{1}{3} \frac{p_\nu + a\sqrt{3\lambda_\nu}}{1 + \sqrt{\lambda_\nu}}$$

thermalization depth: for

$$\tau_\nu \geq \frac{1}{\sqrt{\lambda_\nu}}$$

$$J_\nu \rightarrow B_\nu$$

$J_\nu < B_\nu$ in outer parts
of atmosphere

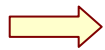


The Milne-Eddington model

compute the line profile

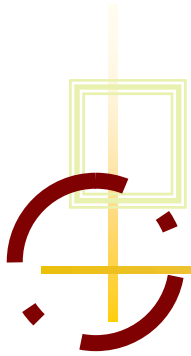
residual flux $R_\nu = \frac{H_\nu(0)}{H_c(0)}$

for H_c : $\beta_\nu = \frac{\kappa_\nu^L}{\kappa_\nu^C} = 0 \implies p_\nu = b \quad \lambda_\nu = 1$



$$H_c(0) = \frac{1}{3} \frac{b + a\sqrt{3}}{2}$$

$$R_\nu = \frac{H_\nu(0)}{H_c(0)} = 2 \frac{p_\nu + \sqrt{3\lambda_\nu}a}{(1 + \sqrt{\lambda_\nu})(b + \sqrt{3}a)}$$



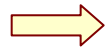
The Milne-Eddington model

a) case $\epsilon = 1$ (LTE: pure absorption lines)

$$S_\nu^L = B_\nu$$

$$S_\nu^c = B_\nu$$

$$\lambda_\nu = \frac{1 + \epsilon\beta_\nu}{1 + \beta_\nu} = 1$$



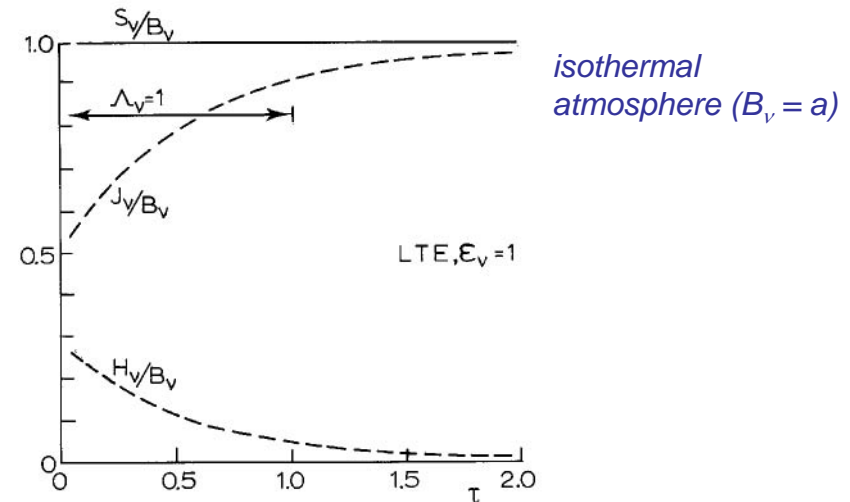
$$R_\nu = \frac{\frac{b}{1+\beta_\nu} + \sqrt{3}a}{b + \sqrt{3}a}$$

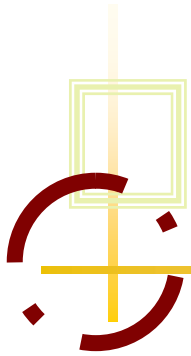
for strong line: $\beta_\nu = \kappa_L / \kappa_c \gg 1$

$$R_\nu = \frac{\sqrt{3}a}{b + \sqrt{3}a} = \frac{B_\nu(0)}{B_\nu(\tau_c = 1/\sqrt{3})} \neq 0$$

emergent flux determined by surface value of Planck function

Non-zero residual intensity





The Milne-Eddington model

in LTE the residual flux is non-zero also for strong absorption lines

e.g. in the Sun $b/a \approx 9/4$ \rightarrow $R = 0.44$ ($\lambda = 5000 \text{ \AA}$)

However resonance lines such as Na D have $R \sim 10^{-3} - 10^{-4}$

b) case $\varepsilon = 0$ (extreme non-LTE: pure scattering lines)

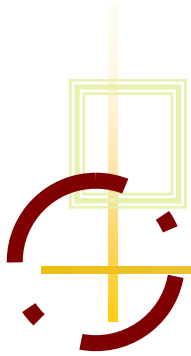
$$\lambda_\nu = \frac{1}{1 + \beta_\nu} \quad R_\nu = \frac{\frac{1}{1 + \beta_\nu} + \sqrt{\frac{3}{1 + \beta_\nu}} a}{(1 + \sqrt{\frac{1}{1 + \beta_\nu}})(b + \sqrt{3}a)}$$

as $\beta_\nu \rightarrow \infty$ (strong line):

$$R_\nu = 0$$

for a strong line formed by scattering

scattering removes all photons \rightarrow no photon emerges from surface

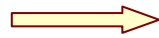


Theoretical curve of growth

standard diagnostic tool for the determination of metal abundances in cool stars
simple model which allows equivalent widths to be calculated analytically

assumptions:

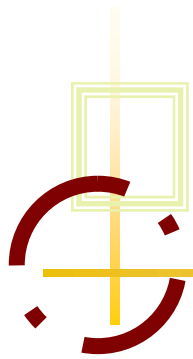
1. pure absorption lines
2. Milne-Eddington model
3. LTE
4. $\varepsilon_\nu = 1$ (no scattering)



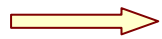
$$R_\nu = \frac{\frac{b}{1+\beta_\nu} + \sqrt{3}a}{b + \sqrt{3}a}$$

$$\beta_\nu = \frac{\kappa_\nu^L}{\kappa^C} = \frac{n_l B_{lu} - n_u B_{ul}}{\kappa^C} \frac{h\nu}{4\pi} \varphi_\nu$$

$$= \frac{n_l B_{lu} (1 - e^{-h\nu/kT})}{\kappa^C} \frac{h\nu}{4\pi} \varphi_\nu = \frac{n_l}{\kappa^C} (1 - e^{-h\nu/kT}) \frac{\pi e^2}{mc} f_{lu} \varphi_\nu = \beta_0 \varphi_\nu$$

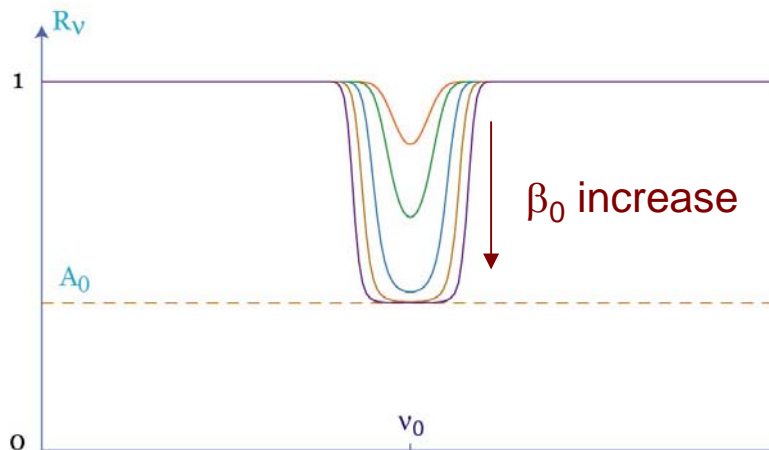


Theoretical curve of growth

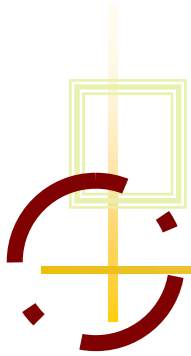


$$R_\nu = \frac{\frac{b}{1 + \beta_0 \varphi_\nu} + \sqrt{3}a}{b + \sqrt{3}a}$$

$$A_\nu = 1 - R_\nu = \frac{\beta_0 \varphi_\nu}{1 + \beta_0 \varphi_\nu} \left(\frac{b}{b + \sqrt{3}a} \right) = A_0 \frac{\beta_0 \varphi_\nu}{1 + \beta_0 \varphi_\nu} \quad \text{line depth}$$



in LTE non-zero central intensity even for strong lines ($\beta_0 \rightarrow \infty$)



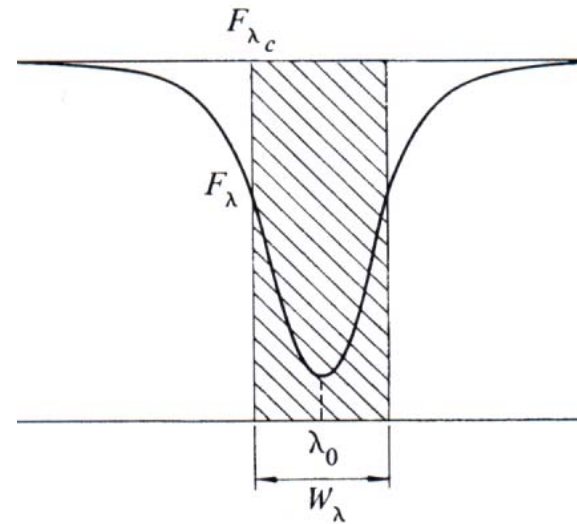
Theoretical curve of growth

Equivalent width

$$W_\nu = \int_0^\infty A_\nu d\nu$$

→

$$W_\nu = A_0 \beta_0 \int_0^\infty \frac{\varphi_\nu}{1 + \beta_0 \varphi_\nu} d\nu$$

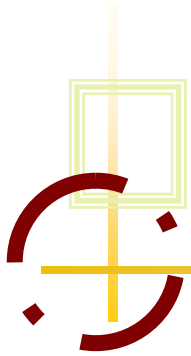


use Voigt profile

$$\varphi(\nu - \nu_0) = \frac{1}{\pi^{1/2} \Delta\nu_D} H(a, v)$$

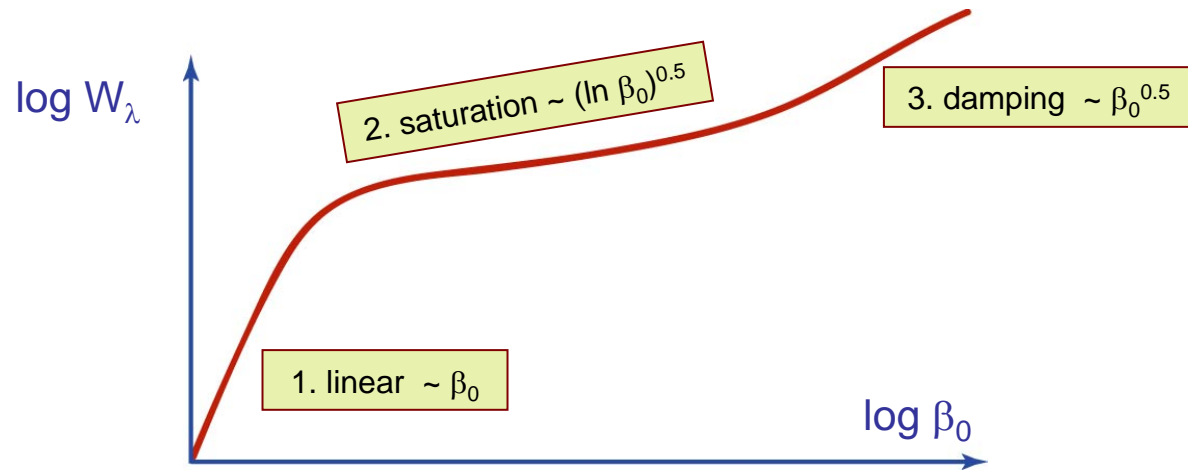
$$\downarrow$$

$$\approx e^{-v^2} + \frac{a}{\pi^{1/2} v^2}$$



Theoretical curve of growth

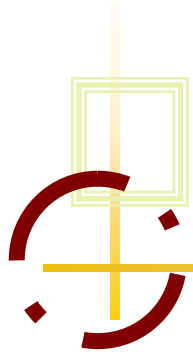
3 regimes



1. linear regime: unsaturated Doppler core $H(a, v) \approx e^{-v^2}$

$$W_\nu \approx \frac{A_0 \beta_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-v^2}}{1 + \frac{\beta_0}{\sqrt{\pi} \Delta \nu_D}} dv = \frac{A_0 \beta_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \left(1 - \frac{\beta_0}{\sqrt{\pi} \Delta \nu_D} e^{-v^2} + \dots \right) dv \approx A_0 \beta_0$$

$\frac{\beta_0}{\Delta \nu_D} < 1$



Theoretical curve of growth

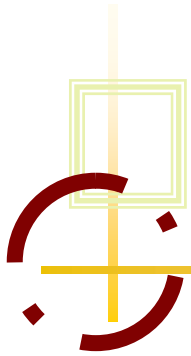
2. saturation part: core saturation but line wings not important

as in previous case but $\frac{\beta_0}{\Delta\nu_D} > 1$

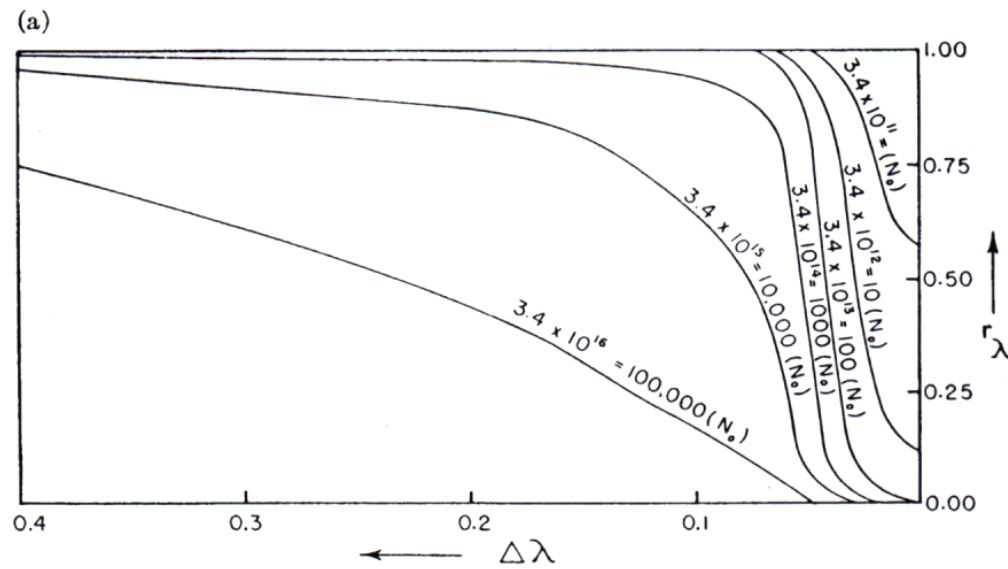
$$W_\nu \approx 2A_0 \Delta\nu_D \sqrt{\ln[\beta_0/(\sqrt{\pi}\Delta\nu_D)]} \sim \sqrt{\ln\beta_0}$$

3. damping part: line wings dominate $H(a, \nu) \approx \frac{a}{\pi^{1/2}\nu^2}$

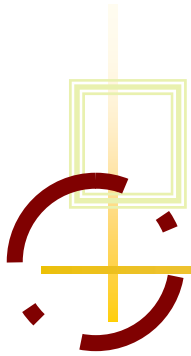
$$W_\nu \approx \frac{A_0\beta_0}{\pi} a \int_{-\infty}^{\infty} \frac{1}{\nu^2 + \frac{\beta_0 a}{\pi\Delta\nu_D}} d\nu = A_0 (a\pi\Delta\nu_D\beta_0)^{1/2}$$



Theoretical curve of growth



effect on a spectral line of the increase of absorbers along the line of sight



Theoretical curve of growth

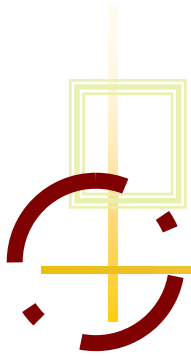
$$\text{In general: } W_\nu = f(\beta_0) \quad \text{or} \quad f\left(\frac{\beta_0}{\Delta\nu_D \sqrt{\pi}}\right) = f(\beta^*)$$

$$\beta^* = \frac{\pi e^2}{mc} f_{lu} \frac{n_l}{\kappa^c} (1 - e^{-h\nu/kT}) \frac{1}{\Delta\nu_D \sqrt{\pi}}$$

$$\kappa_\nu^c = \kappa_0^c (1 - e^{-h\nu/kT}) \quad \text{in LTE}$$

$$n_l = n_1 \frac{g_l}{g_1} (1 - e^{-h\nu/kT}) \quad \text{Boltzmann}$$

$$\Delta\nu_D = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}} = \frac{1}{\lambda} \sqrt{\frac{2kT}{m}}$$

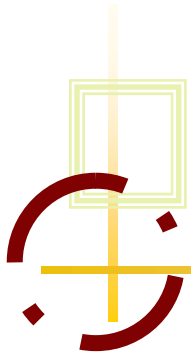


Theoretical curve of growth

$$\log \beta^* = \log(g_l f_{lu} \lambda) + \log e^{-E_{1l}/kT} + \log \left(\frac{n_1}{g_1 \kappa_0^c} \frac{\sqrt{\pi} e^2}{mc} \sqrt{\frac{m}{2kT}} \right)$$

$$\log \beta^* = \log(g_l f_{lu} \lambda) - \frac{5040 E_{1l}}{T} + \log C$$

1. for a single ionization stage $C = \text{const}$
2. lines belonging to one ionization stage should form a curve of growth: β^* varies as a function of line transition
3. if T and κ_0 known: shift observed W_ν curve until it matches theoretical curve
4. from n_1 calculate total abundances using Saha-Boltzmann equations

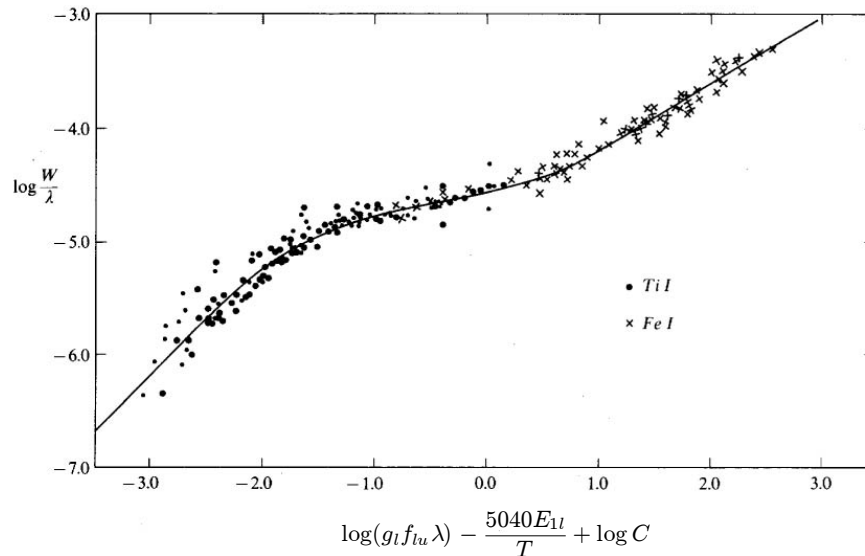


Theoretical curve of growth

empirical curve of growth

plot $\log (W/\lambda)$ vs $\log (gf\lambda) - 5040 E_{11}/T$ for each line

adjust T to minimize scatter around a mean curve (excitation temperature typical of line formation region)



*empirical curve of growth for
iron and titanium lines in the Sun*