Astronomy 241: Review Problems #1

Review the questions below, and be prepared to discuss them in class on September 30 and October 2. Modified versions of some of these questions will be used in the midterm exam on October 9.

1. State Kepler’s three laws and give an example of how each one is used.

2. The International Space Station is in a circular orbit with a radius of \( a_{\text{ISS}} = 6800 \) km. You are in a spaceship in the same orbit as the ISS, but trailing it by a distance of 100 km.
   (a) What happens if you try to catch up by firing your rocket engines in the direction of your orbit?
   (b) How would you actually fire your rockets in order to reach the ISS?

3. The Moon orbits the Earth with an orbital period \( P_M = 27.3 \) day and a semi-major axis \( a_M = 3.84 \times 10^5 \) km. Using this information, calculate
   (a) the semi-major axis \( a_{\text{geo}} \) of a geosynchronous orbit \( (P_{\text{geo}} = 1 \) day), and
   (b) the orbital period \( P_{\text{leo}} \) of a low-Earth orbit \( (a_{\text{leo}} = 6600 \) km).

4. Bananas are among the most radioactive foods (though still safe) due to high potassium concentration. Consider a 0.2 kg banana that contains \( \sim 4 \times 10^{-4} \) kg of potassium. Approximately \( 1.2 \times 10^{-6} \) of that potassium is \( ^{40}\text{K} \), which decays with an energy of 33.5 MeV = \( 5.37 \times 10^{-12} \) kg m\(^2\) s\(^{-2}\) per nucleus. The half-life of \( ^{40}\text{K} \) is \( \tau_0 \approx 1.25 \times 10^9 \) yr.
   (a) Estimate the number of \( ^{40}\text{K} \) atoms in a typical banana today, given that a \( ^{40}\text{K} \) atom has \( \sim 40 \) times the mass \( m_H \approx 1.67 \times 10^{-27} \) kg of a hydrogen atom.
   (b) Evaluate the energy released when the \( ^{40}\text{K} \) decays entirely.
   (c) Write an expression for the number of \( ^{40}\text{K} \) atoms remaining in the banana as a function of time. (As in most physics problems, you may ignore other complicating processes such as rotting that will alter the banana with time.)
   (d) Write an expression for the decay rate of \( ^{40}\text{K} \) atoms (the number of atoms decaying per second) and evaluate it at the present time.
(e) Using your result for (d), evaluate the power (in watts) from the decay $^{40}\text{K}$ at the present time.

5. Given that seawater is incompressible with a density of $1025 \text{ kg m}^{-3}$, how deep do you need to dive before the total pressure (ocean plus atmosphere) is $2P_0$, where the typical pressure at the Earth’s surface is $P_0 = 1.013 \times 10^5 \text{ kg m}^{-1} \text{ s}^{-2}$?

6. The Moon has a mass of $M_M = 7.3 \times 10^{22} \text{ kg}$ and radius $R_M = 1.74 \times 10^6 \text{ m}$ and was formed within $\sim 100$ Myr of the formation of the Earth. During the “Late Heavy Bombardment” several 100 Myr after that, the Moon was struck with $\sim 1 \times 10^{18} \text{ kg}$ of material that arrived as many impactors, each adding material to the surface of the Moon. What was the total energy released by this accretion of material?

7. Transferring material between Mars and Earth with a “sample return” mission is sometimes envisioned as the following separate rocket maneuvers:

   \text{i}) Liftoff from the surface of Mars to low-Mars orbit ($\sim 200 \text{ km}$ elevation);
   
   \text{ii}) A rocket burn out of Mars orbit and into an “Hohmann transfer orbit” bound for Earth. This orbit is elliptical, intersecting the orbits of Mars and Earth at either end of the semi-major axis. (Assume that the orbits of Mars and Earth are circular.)
   
   \text{iii}) A rocket burn out of the Hoffman transfer orbit and into low-Earth orbit.
   
   \text{iv}) A landing sequence that safely delivers the satellite from low-Earth orbit to the surface.

   (a) What is the orbital period $P$ of low-Mars orbit, given the size of Mars, $R_{Ma} = 3400 \text{ km}$?
   
   (b) How long does it take the spacecraft to travel from from Mars to Earth on the Hohmann transfer orbit? You can approximate the semi-major axis of Mars as 1.5 AU.
   
   (c) What are the orbital speeds at pericenter and apocenter of the Hohmann transfer orbit?

8. The dwarf planet Sedna has a radius $R = 1000 \text{ km}$, an albedo $A = 0.32$, an apastron distance $a_{apo} = 937 \text{ AU}$, and an eccentricity $e = 0.857$.

   (a) Assuming that Sedna has the same density as Pluto ($\rho = 2000 \text{ kg m}^{-3}$), what is the mass of Sedna?
   
   (b) What is Sedna’s semi-major axis in AU?
(c) What is the equilibrium temperature of Sedna at periastron and apastron? (The Sun has radius $R_{\odot} \simeq 7.0 \times 10^8$ m and temperature $T_{\odot} \simeq 5800$ K.)

(d) What is the scale height of a possible H$_2$O atmosphere on Sedna? Note that the mass of an H$_2$O molecule is approximately $\sim 18$ times the mass $m_{\text{H}} \simeq 1.67 \times 10^{-27}$ kg of a hydrogen atom.

(e) Find the temperature required for H$_2$O molecules to escape Sedna’s putative atmosphere?

9. An asteroid of mass $m \ll M_\oplus$ approaches the Earth on a hyperbolic trajectory as shown in Fig. 1. This figure also defines the initial velocity $v_0$ and “impact parameter” $b$. Ignore the gravity of the Moon or other solar system objects.

(a) What are the energy $E$ and angular momentum $L$ of this orbit? (Hint: one is easily evaluated when the asteroid is far away, the other when it makes its closest approach.)

(b) Find the orbital eccentricity $e$ in terms of $M_\oplus$, $E$, $L$, and $m$ (note that $e > 1$). Then use your results for part (a) to express $e$ in terms of $M_\oplus$, $v_0$, and $b$.

(c) What is the actual pericentric separation $r_p < b$ of the asteroid’s orbit?

(d) Long after this encounter, the asteroid will again be moving in a straight line with velocity $v_0$. What is the angle $\theta$ between this line and the asteroid’s initial trajectory?

10. Three of Jupiter’s moons are locked into a resonance with the orbital periods of Io, Europa, and Ganymede having ratios of

$$P_{\text{Io}} : P_{\text{Eu}} : P_{\text{Ga}} = 1 : 2 : 4$$

(1)
In other words, Io completes four orbits, and Europa completes two, in the time it takes Ganymede to complete one. What is the ratio of their orbital radii $a_{Io} : a_{Eu} : a_{Ga}$?

11. We think that Mercury originally had an iron core comprising $\sim 30\%$ of its mass (much like the Earth). After Mercury differentiated, a giant impact blasted much of the rocky mantle away without seriously disturbing the mass of the core; as a result, Mercury’s core now makes up $\sim 60\%$ of the planet’s mass.

(a) Given that Mercury currently has mass $M_{Me} = 3.3 \times 10^{23}$ kg, what was its mass before the impact? You may ignore the mass of the impacting body.

(b) Roughly how much energy would have been required to eject the mantle material Mercury lost?

(c) Assuming the impacting body had a velocity of 50 km s$^{-1}$, how massive would it have had to be to deliver the necessary kinetic energy? Compare your result with Mercury’s original mass.

12. Consider a homogeneous planet (i.e., one with constant internal density $\rho$) of mass $M$ and radius $R$.

(a) Show that at radius $r \leq R$, the gravitational acceleration is

$$g(r) = \frac{GM_r}{R^3}$$  \hspace{1cm} (2)

(b) Use the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -\rho g(r)$$  \hspace{1cm} (3)

to find an expression for the central pressure, $P_c$.

13. Consider a homogeneous planet (i.e., one with constant internal density $\rho$) of mass $M$ and radius $R$. Show that the gravitational energy released during its formation is

$$E = \frac{3GM^2}{5R}$$  \hspace{1cm} (4)
14. The four long-lived radioactive isotopes which currently heat Earth’s interior are $^{238}U$, $^{235}U$, $^{232}Th$, and $^{40}K$. The table below gives the half-life $\tau_{1/2}$ of each isotope (in units of $10^9$ yr) and the power per unit mass of mantle rock $\epsilon_{\text{now}}$ each isotope currently generates (in units of $10^{-12}$ W kg$^{-1}$ = $10^{-12}$ m$^2$ s$^{-3}$).

<table>
<thead>
<tr>
<th></th>
<th>$^{238}U$</th>
<th>$^{235}U$</th>
<th>$^{232}Th$</th>
<th>$^{40}K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{1/2}$</td>
<td>4.5</td>
<td>0.70</td>
<td>14.0</td>
<td>1.25</td>
</tr>
<tr>
<td>$\epsilon_{\text{now}}$</td>
<td>2.91</td>
<td>0.125</td>
<td>3.27</td>
<td>1.08</td>
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(a) Which of these isotopes made the greatest contribution to Earth’s internal heating 4.6 $\times$ $10^9$ yr ago?

(b) Calculate the total $\epsilon$ at that time.

(c) What is the total energy (per kilogram) that will be released by these four isotopes from the birth of the solar system 4.6 $\times$ $10^9$ yr ago to the far future ($t \rightarrow \infty$)?

15. A greenhouse skeptic argues “The ‘greenhouse gasses’ in the Earth’s atmosphere are cooler than the surface of the Earth. The Second Law of Thermodynamics says that heat can never flow from cold to hot. Thus it’s impossible for greenhouse gasses to warm the Earth.” Your response?

16. Suppose the Moon’s mass doubled, with no immediate change in its present orbit.

(a) How much would the height of tides on Earth change as a result?

(b) How would the precession of the Moon’s orbit change as a result?

(c) How would the precession of the Earth’s rotation change as a result?

(d) How would the gradual expansion of the Moon’s orbit change as a result?

(e) How would the gradual decrease in the Earth’s rate of rotation change as a result?