Astronomy 241: Problem Set #7
Due Thursday, October 30, 2014

Solve the problems listed below and write up your answers clearly and completely. Do not turn in rough work; instead, make a clean copy after checking your calculations. Use English sentences and phrases to explain your solution and describe key equations. Show your work!

Collaboration is encouraged. However, you should attempt each problem yourself first and write up your work independently. Copied solutions will receive no credit.

1. Assume the Oort Cloud has a total mass \( M_C \approx 100 M_\odot \approx 6 \times 10^{26} \text{ kg} \) and contains \( N_C \approx 10^{13} \) objects with densities \( \bar{\rho} \approx 1000 \text{ kg m}^{-3} \).
   (a) Assuming that Oort Cloud objects are roughly spherical, what is the radius \( R_0 \) of an average Oort Cloud object?
   (b) Suppose \( N_C \) such objects are uniformly distributed through a spherical volume of radius \( R_C = 10^4 \text{ AU} \). What is the mean free path \( l \) of a typical Oort Cloud object?
   (c) Assume a typical Oort Cloud object has a roughly circular orbit with a semi-major axis \( a = 10^4 \text{ AU} \). How many orbits does it complete in \( 4.5 \times 10^9 \text{ yr} \)?
   (d) Using your results for parts (b) and (c) above, what is the probability that a typical Oort Cloud object has been involved in a collision since the Solar System formed?

2. Saturn’s rings have nonzero thickness because ring particle orbits are slightly inclined with respect to each other. Thus, a particle which attains a maximum height \( z = h \) above the ring plane at one point on its orbit will descend to a minimum height \( z = -h \) below the ring plane at another point. Saturn’s A ring has a radius of \( r_A \approx 1.30 \times 10^8 \text{ m} \), and Saturn’s mass is \( M_S \approx 5.7 \times 10^{26} \text{ kg} \).
   (a) Assuming for simplicity that Saturn’s gravitational field is spherical, what is period of the vertical oscillation executed by a ring particle orbiting at radius \( r = r_A \)?
   (b) Given that Saturn’s A ring is about 30 m thick, estimate the average vertical speed of a typical ring particle.
   (c) Ring particles are chunks of ice; most are a few cm in diameter. Suppose two ring particles collide at the speed you just calculated. Would you expect them to shatter, melt, stick together gravitationally, or rebound elastically?

3. A nearly-circular orbit of radius \( r \) around an oblate planet of mass \( M \), mean radius \( R \), and second gravitational moment \( J_2 \) has orbital, radial, and vertical periods

\[
P(r) \approx 2\pi \sqrt{\frac{r^3}{GM}} \left(1 + \frac{3}{2} J_2 \frac{R^2}{r^2} + \cdots \right)^{-1/2},
\]

\[
P_r(r) \approx 2\pi \sqrt{\frac{r^3}{GM}} \left(1 - \frac{3}{2} J_2 \frac{R^2}{r^2} - \cdots \right)^{-1/2},
\]

\[
P_z(r) \approx 2\pi \sqrt{\frac{r^3}{GM}} \left(1 + \frac{9}{2} J_2 \frac{R^2}{r^2} + \cdots \right)^{-1/2},
\]

respectively. (For this problem, it’s easier to think in terms of period \( P \) instead of angular velocity \( 2\pi/P \), but equations (1) – (3) mean the same thing as the expressions for the orbital, radial, and vertical angular velocities \( n(r) \), \( \kappa(r) \), and \( \mu(r) \) you’ve seen in class.)
(a) Jupiter has mass $M_J \simeq 1.90 \times 10^{27}$ kg, mean radius $R_J \simeq 7.0 \times 10^7$ m, and gravitational moment $J_2 \simeq 0.015$. Calculate $P_r$, $P_z$, and $P$ for Jupiter’s satellite Amalthea, which orbits at radius $r_A = 1.81 \times 10^8$ m. Verify that $P_z < P < P_r$.

(b) Amalthea travels exactly 360° around Jupiter (one orbit) in time $P$. How many degrees does it travel around Jupiter in time $P_z$? The difference between this angle and 360° tells you how much the line of nodes (intersection of orbit and equatorial planes) precesses per orbit. If the orbit is counter-clockwise, which way does the line of nodes shift?

(c) Repeat parts (a) and (b) for a hypothetical satellite orbiting at radius $r = 2r_A$. Measured in terms of degrees per orbit, how much faster or slower does this satellite’s orbit precess?