Solve the problems listed below and write up your answers clearly and completely. Do not turn in rough work; instead, make a clean copy after checking your calculations. Use English sentences and phrases to explain your solution and describe key equations. Show your work!

Collaboration is encouraged. However, you should attempt each problem yourself first and write up your work independently. Copied solutions will receive no credit.

1. A spherical shell of matter with density $\rho$, radius $R$, and thickness $dR$ has moment of inertia

$$dI = \frac{8\pi}{3} \rho R^4 dR$$

(a) Show that a homogeneous (constant density) sphere of mass $M$ and radius $R$ has moment of inertia

$$I_h = \frac{2}{5} MR^2$$

(b) Evaluate $I_h$ for a sphere with the mass $M = M_{\oplus} \simeq 6.0 \times 10^{24}$ kg and radius $R = R_{\oplus} \simeq 6.4 \times 10^6$ m of the Earth.

(c) We can approximate the Earth as consisting of a dense core of radius $R_c \simeq 3.5 \times 10^6$ m containing 31% of the Earth’s mass, and a lower-density mantle containing the remaining 69% of the Earth’s mass. Evaluate the moment of inertia $I$ for this inhomogeneous configuration. Compare this answer with the one you got for part (b).

(d) Suppose that the Earth formed as a homogenous sphere with $M = M_{\oplus}$ and radius $R = R_{\oplus}$ and then gravitationally settled to the inhomogeneous configuration described in (c). How much energy ($E_{\text{therm}}$) was released in this settling? You can ignore compression at high pressure in your calculation (i.e., assume that $R$ is constant during the settling).

(e) If the Earth has a specific heat $C \simeq 3k_B/40m_H \simeq 600$ m$^2$s$^{-2}$K$^{-1}$, what is the increase in temperature $\Delta T$ from $E_{\text{therm}}$?

2. The geothermal heat flux escaping from the interior of the Earth is $F_{\text{geo}} \simeq 0.09$ W m$^{-2}$.

(a) In the absence of solar heating, the Earth’s surface would eventually cool to a temperature $T_{\text{geo}}$ such that it emits exactly $F_{\text{geo}}$ in the form of black-body radiation. Calculate $T_{\text{geo}}$.

(b) To get rid of the heat it absorbs from the Sun, the Earth radiates like a black body with a temperature $T_{\text{eq}} \simeq 255$ K. Compare the power radiated by each square meter of the Earth’s surface to the geothermal flux.

(c) Billions of years ago, the geothermal flux could have been an order of magnitude greater than it is today. Estimate how much warmer the Earth’s surface would need to be to get rid of this additional heat.

3. $^{238}\text{U}$ decays into $^{206}\text{Pb}$, releasing 51.7 MeV = $8.28 \times 10^{-12}$ kg m$^2$s$^{-2}$ of energy per nucleus. The half-life of $^{238}\text{U}$ is $\tau_0 \simeq 4.6 \times 10^9$ yr.

(a) Given that a $^{238}\text{U}$ atom has $\sim 238$ times the mass $m_H \simeq 1.67 \times 10^{-27}$ kg of a hydrogen atom, evaluate the total amount of energy released when 1 kg of pure $^{238}\text{U}$ decays entirely into $^{206}\text{Pb}$.

(b) Assume you have 1 kg of pure $^{238}\text{U}$ at time $t = 0$. Derive an expression for the number of $^{238}\text{U}$ nuclei remaining at time $t$.

(c) Using your result for (b), evaluate the power (in watts) released by 1 kg of pure $^{238}\text{U}$.